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Decay of momentum flux in submerged jets

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Slender laminar and turbulent, plane and axisymmetric jets emerging from orifices in plane or conical walls are studied at large distances from the orifices. The entrainment of momentum coupled with the entrainment of volume into a jet is determined, and its effect on the flow field is analysed by combining inner and outer expansions with a multiple scaling approach.

In turbulent (plane or axisymmetric) jets, the axial velocity decreases more rapidly than predicted by classical boundary-layer solutions, and the momentum flux vanishes as the distance from the orifice tends to infinity. The analysis unveils a source of discrepancies in previous experimental data on turbulent jets.

In a laminar plane jet, the momentum flux changes but little. In a laminar axisymmetric jet, the momentum flux changes slowly, yet considerably. When a critical distance from the orifice is approached, the momentum flux in the jet becomes very small, the jet diameter very large, and a toroidal viscous eddy is predicted. The structure of the flow field is briefly discussed.

1. Introduction

The classical self-similar solutions of incompressible jet flows (Schlichting 1933; Bickley 1937; Tollmien 1926; Görtler 1942; cf. also Schlichting 1979) are asymptotic solutions based on the following main assumptions (figure 1):

(*a*) The jet is very slender. This implies large values of the jet Reynolds number (as defined below) in the laminar case, while the width of turbulent jets is known to be independent (or nearly independent) of the jet Reynolds number. Applying boundary-layer theory, the flow field is divided in (at least) two regions, i.e. the jet boundary layer and the outer flow induced by the entrainment of fluid into the jet. (In the presence of a wall there may – or may not – be an additional boundary layer at the wall, cf. §2.)

(*b*) The distance r from the orifice, from which the jet emerges with mean-square velocity u_0, is very large compared with the width or diameter, $2a_0$, of the orifice. Thus the volume flux from the orifice is neglected, and the flow at r does not depend on the parameters u_0 and a_0 separately but depends only on the kinematic momentum flux which is given by, respectively, $M_0 = 2a_0 u_0^2$ for a plane (two-dimensional) jet and $2\pi M_0 = \pi a_0^2 u_0^2$ for an axisymmetric jet.

(*c*) In the absence of volume forces, the momentum flux in the jet boundary layer is conserved, i.e.

$$M = (2-j)\int_0^\infty u^2 Y^j \,\mathrm{d}Y = M_0 = \text{const},$$

where u is the velocity component in direction of the jet axis, Y is the lateral boundary-layer co-ordinate, and $j = 0$ or 1 for plane and axisymmetry flow, respectively.

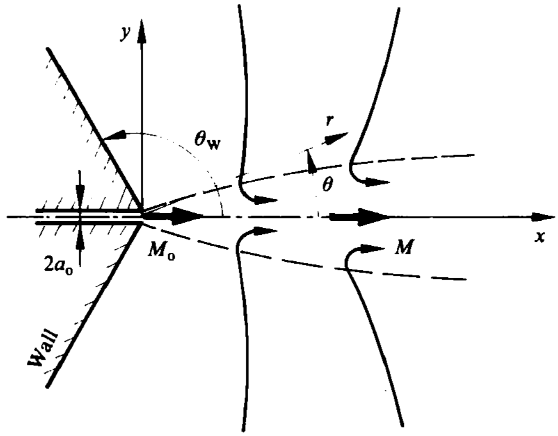

FIGURE 1. Sketch of flow field, coordinate systems and parameters.

Although each of the assumptions may be subject to criticism (Schneider 1983), we shall retain (*a*) and (*b*) but investigate deviations from (*c*) below. The investigations are motivated by a number of peculiar results which have been found in previous work on both laminar and turbulent jets.

For laminar jets with large Reynolds number, the method of matched asymptotic expansions was applied to find second-order boundary-layer solutions (Rubin & Falco 1968; Mitsotakis, Schneider & Zauner 1984). In the presence of walls, the asymptotic behaviour at large distances from the orifice turned out to be essentially different for plane and axisymmetric jets, respectively. While for plane jets the expansion is regular as $r \rightarrow \infty$ (Rubin & Falco 1968), the second-order terms in the axisymmetric problem grow beyond bounds as $r \rightarrow \infty$ (Mitsotakis *et al.* 1984). This is associated with a logarithmic (i.e. unlimited) decay of the momentum flux in the jet, and it severely restricts the applicability of boundary-layer theory to laminar axisymmetric jets emerging from walls. The failure of the boundary-layer analysis also raises the question of what the flow field looks like at very large distances from the orifices.

Regarding turbulent jets, the assumption of constant momentum flux is in apparent agreement with some experiments (Bradbury 1965) whereas other measurements reveal discrepancies between theory and experiment. Miller & Comings (1957), for instance, observed a centreplane velocity decay slightly larger than predicted by classical theory. As the jet width was obtained in agreement with theory, the experiments by Miller & Comings (1957) suggest a slow decrease of the momentum flux through the jet cross-section with increasing distance from the orifice. With respect to the present analysis it will be of importance to note that Miller & Comings (1957) performed their experiments with a nozzle located in a wall perpendicular to the jet axis, whereas there was no such wall in Bradbury's (1965) experiments. The problem became even more controversial when Hussain & Clark (1977) concluded from their data that the 'total' momentum flux (including the mean momentum flux due to velocity fluctuations) increases with increasing distance, and is not completely balanced by the mean static pressure integral.

From a theoretical point of view, the conservation law for the momentum flux was examined by Kraemer (1971) and Kotsovinos (1978). While Kraemer studied global momentum balances, Kotsovinos estimated (without recourse to an asymptotic analysis) the various terms in the momentum balance equation, and concluded that for a plane turbulent jet out of a wall the momentum flux in the jet is reduced due to the induced outer flow. However, as the distance from the orifice tends to infinity, Kotsovinos (1978) predicts negative-infinite momentum flux, i.e. the approximation breaks down. The failure is due to the fact that Kotsovinos (1978), like Kraemer (1971), did not account for the reactive effect which the decrease of the momentum flux exerts on the induced flow.

In the present paper the problem of jet flows at large distances from an orifice in a wall is studied by coupling the jet flow with the induced outer flow. This can be done in various ways. In the main paper the momentum flux in the jet boundary layer is considered to be a slowly varying function of the distance, and the coupling is established by accounting for the momentum flux from the induced outer flow into the jet (entrainment of momentum). A more formal approach is given in Appendix B, where the method of multiple scales is combined with inner and outer expansions. For laminar axisymmetric flow, the analysis not only predicts large deviations from classical (as well as second-order) boundary-layer theory, but also indicates an unexpected breakdown of the slender-jet assumption. This suggests a flow field quite different from the common model of laminar jets. For turbulent (plane or axisymmetric) flow, the analysis provides answers to the questions concerning the invariance of the momentum flux, and unveils the source of previous discrepancies.

2. Entrainment of volume and momentum

Consider a plane jet issuing from a corner into the plane of symmetry, or an axisymmetric jet issuing from the tip of a conical wall into the direction of the axis of symmetry. The walls are assumed to be infinite in extent, the apex angle is $2\Theta_w$ (figure 1). If $\Theta_w = \frac{1}{2}\pi$ we have the particular case of a jet emerging from an infinite plane wall perpendicular to the jet axis.

We shall distinguish between flow types with the help of an integer j, where $j = 0$ for plane (two-dimensional) flow, and $j = 1$ for axisymmetric flow. Let $(2\pi)^j\, V$ be the volume flux through a cross-section of the jet, and $(2\pi)^j\, M$ the kinematic momentum flux, i.e. the momentum flux divided by the constant density of the fluid. Note that both V and M are in terms of unit azimuthal angle of axisymmetric flow. A (local) jet Reynolds number is defined by $Re_x = (Mx)^{\frac{1}{2}}\nu^{-1}$ for plane flow, and $Re = M^{\frac{1}{2}}\nu^{-1}$ for axisymmetric flow, where x is the axial distance from the jet origin (cf. figure 1), and ν is the (constant) kinematic viscosity of the fluid.

It is well known that V increases with increasing x. Classical similarity considerations (with M = const, cf. Schlichting 1979) yield

$$\left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)^2 = \epsilon M x^{j-1} \tag{1}$$

with an entrainment coefficient ϵ as given in table 1. Note that $\epsilon \to 0$ as $Re_x \to \infty$ or $Re \to \infty$ in the laminar cases, while ϵ is a numerically small constant for turbulent jets. This justifies the assumption of a slender jet and admits a boundary-layer approach.

Entrainment of volume (or mass) into a jet gives rise to a flow of the ambient fluid (figure 1). To a first approximation, the slender jet acts as a line sink to the outer flow. For two-dimensional flow, the outer flow is, again to a first approximation, an inviscid potential flow. Reichardt (1942), Taylor (1958), Wygnanski (1964), Rubin

	Plane	Axisymmetric
Laminar	$\left(\frac{4}{3Re_x}\right)^{\frac{2}{3}}$	$\frac{16}{Re^2}$
Turbulent	0.085†	0.013‡

† According to Tollmien (1926); Reichardt (1942).
‡ According to Ricou & Spalding (1961).

TABLE 1. Entrainment coefficient ϵ as defined by (1)

& Falco (1968) as well as Kraemer (1971) calculated such flows with and without walls. For what follows it suffices to conclude from the results that the streamlines of the outer flow enter the (very slender) jet at angles $\frac{1}{3}\Theta_w$ and $\frac{1}{2}\Theta_w$ for laminar and turbulent flow, respectively.

For axisymmetric flow, the problem of predicting the induced outer flow is more complex. Neglecting viscosity in the outer flow is incorrect for laminar axisymmetric jets (no matter how large the jet Reynolds number Re may be) (Squire 1952; Potsch 1981; Schneider 1981). Thus the full Navier–Stokes equations have to be solved in order to find the outer flow field. Appropriate boundary conditions are the non-slip condition at the wall, and a condition which accounts for the line sink of volume at the axis. For a conical wall, the variables can be separated in terms of spherical coordinates r, Θ (figure 1) by writing Stokes' stream function ψ as

$$\psi = 4\nu r f(\xi), \quad \xi = \tfrac{1}{2}(1-\cos\Theta). \tag{2}$$

For $f(\xi)$, the following boundary-value problem has been obtained and solved numerically (Schneider 1981)

$$\xi(1-\xi)f' - (1-2\xi)f + f^2 = -2C^2\xi(1-\xi/\xi_w); \tag{3a}$$

$$C^2 = -[1+f'(0)] > 0; \tag{3b}$$

$$f(\xi_w) = 0, \tag{3c}$$

where

$$\xi_w = \tfrac{1}{2}(1-\cos\Theta_w).$$

If the axisymmetric jet flow is turbulent, the induced flow may be approximated by an inviscid one if the Reynolds number is sufficiently large, say $Re > 5000$ (Schneider 1981). For simplicity, we shall assume the latter condition satisfied in what follows. In this case, the outer flow field can be found in closed form. In agreement with Taylor (1958), the result is

$$\psi = r\left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)\left(1-\frac{\xi}{\xi_w}\right).$$

The entrainment of volume by the jet is associated with an entrainment of momentum. This may be seen as follows. Let the jet be surrounded by a control surface whose lateral extension a_c shrinks to zero as $\epsilon \to 0$ (figure 2). Convective flux of axial momentum through the control surface, as well as pressure and viscous stresses at the surface, contribute, in general, to the change of kinematic momentum flux, $\mathrm{d}M/\mathrm{d}x$, in the jet. Using the previous results for the outer flows, the change of kinematic momentum flux in the jet can be written as

$$\frac{\mathrm{d}M}{\mathrm{d}x} = -\frac{1}{2}\left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)^2\frac{m}{x^j}, \tag{4}$$

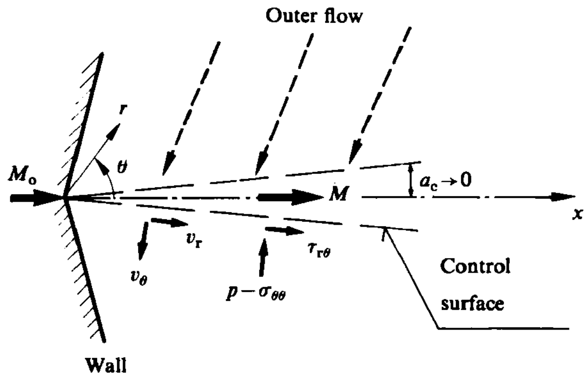

FIGURE 2. Control surface surrounding a slender jet.

	Convection		Pressure	Viscous stresses		
	$v_\theta v_r$	v_θ^2	p	$\sigma_{\theta\theta}$	$\tau_{r\theta}$	m (total)
Plane						
Laminar	$\cot\frac{1}{3}\Theta_w$	—	—	—	—	$\cot\frac{1}{3}\Theta_w$
Turbulent	$\cot\frac{1}{2}\Theta_w$	—	—	—	—	$\cot\frac{1}{2}\Theta_w$
Axisymmetric						
Laminar	$-f'(0)$	-2	$+1$	$+1$	-1	C^2
Turbulent	$[\sin\frac{1}{2}\Theta_w]^{-2}$	-2	$+1$	—	—	$[\cot\frac{1}{2}\Theta_w]^2$

TABLE 2. Terms contributing to the constant m that characterizes the decay of kinematic momentum flux in a slender jet

Θ_w	60°	90°	120°	175°
$f'(0)$	-6.03	-2.91	-1.83	-1.111
C^2	5.03	1.91	0.83	0.111

TABLE 3. The constants $f'(0)$ and C^2 for the outer flow induced by axisymmetric laminar jets, with various wall semi-apex angles Θ_w

where m is a constant which, together with the various contributions to it, is given in table 2. The convective momentum flux through the control surface consists of contributions from $v_\Theta v_r$ and v_Θ^2, respectively, where v_r, v_Θ are the velocity components in (r, Θ)-directions. As $\epsilon \to 0$, the contributions due to v_Θ^2 as well as the pressure contribution are negligibly small in the case of two-dimensional flow. Furthermore, viscous normal and shear stresses, $\sigma_{\Theta\Theta}$ and $\tau_{r\Theta}$, are to be taken into account only if the outer flow is a viscous one, but they cancel out even in this case. The constants $f'(0)$ and $C^2 = -[1+f'(0)]$ are to be obtained from solving the boundary-value problem (3*a*–*c*). Some numerical results are given in table 3. As $\Theta_w \to \pi$, the constant C^2 vanishes and, according to table 2 and (4), the momentum flux in the jet becomes independent of the distance x. This is the particular case of a jet flow in the absence of any walls, as given by Landau's exact similarity solution for a point source of momentum (Rosenhead 1963).

From the point of view of a global momentum balance, the decay of momentum flux in the jet has its counterpart in a force due to pressure and viscous stresses acting on the wall. We shall come back to that point in §6.

3. Turbulent jets with slowly varying momentum flux

As mentioned in the introduction, coupling of turbulent jet flow and induced flow is necessary in order to avoid a breakdown of the analysis as $x \to \infty$. With $\epsilon \ll 1$, it follows from (4), together with (1), that the momentum flux in the jet changes but slowly with the axial co-ordinate. Following the ideas of the method of multiple scales, we introduce a 'slow' variable $X = (x/a_0)^\epsilon$ in addition to the ordinary ('fast') variable x/a_0. Assuming (1) to be locally valid with $M = M(X)$, we may use (1) to eliminate $\mathrm{d}V/\mathrm{d}x$ in (4). (A more formal analysis is given in Appendix A by means of an asymptotic expansion in terms of ϵ.) This yields a differential equation for $M(X)$, which can easily be integrated. After re-introducing x instead of X, one obtains

$$\frac{M}{M_0} = \left(\frac{x_0}{x}\right)^{\frac{1}{2}\epsilon m}, \tag{5}$$

where, according to table 2, $m = [\cot \frac{1}{2}\Theta_w]^{1+j}$. M_0 is the given value of M at the orifice of width or diameter $2a_0$, and x_0 is a constant of integration (with dimension of length), which has been introduced such that $M = M_0$ at $x = x_0$. As M changes but slowly with x, x_0 is supposed to be of the order of a_0, i.e. $x_0/a_0 = \text{const} = O(1)$. The constant cannot be determined within the framework of an asymptotic theory for large values of x/a_0 (see §1 assumption (*b*)). Note, however, that the value of the constant x_0/a_0 does not matter in the present analysis, since replacing x_0 by a_0 in (5) introduces only an error that is small of higher order.

For moderately large distances, which satisfy the relation $\ln(x/x_0) \ll 1/m\epsilon$, (5) may be expanded in terms of the small exponent. The two-term expansion

$$\frac{M}{M_0} = 1 - \epsilon \tfrac{1}{2} m \ln\left(\frac{x}{x_0}\right) + \ldots, \tag{6}$$

agrees with an approximate equation given by Kotsovinos (1978), except for the constants which have different values. In contrast to Kotsovinos' approximation, however, the full equation (5) is applicable to arbitrary large distances from the orifice, and it properly yields $M/M_0 \to 0$ in the limit $x/x_0 \to \infty$.

With the local momentum flux given by (5), the local volume flux can be determined from (1). Neglecting a term $O(\epsilon)$ in the coefficient, one obtains

$$V = (1+j)(\epsilon M)^{\frac{1}{2}} x^{1/(1+j)}. \tag{7a}$$

Since momentum flux and volume flux are proportional to $u_m^2 a^{1+j}$ and $u_m a^{1+j}$, respectively, with jet half-width or radius a, and with velocity u_m at the axis, it follows from (5) and (7*a*) that

$$a = K_a \epsilon^{1/(1+j)} x, \tag{7b}$$

$$u_m^2 = K_u \frac{M_0}{a_0^{1+j}} \left(\frac{x_0}{x}\right)^{1+j+\frac{1}{2}\epsilon m}, \tag{7c}$$

where K_a and K_u are constants of order 1. While the classical result of linear growth of jet width or jet radius with increasing distance is re-obtained with (7*b*), (7*c*) differs from the classical solution by the term $\frac{1}{2}\epsilon m$ in the exponent.

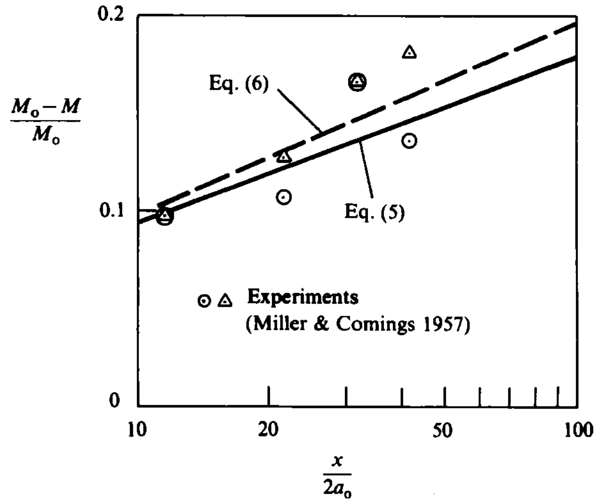

FIGURE 3. Momentum flux in a plane turbulent jet versus distance from the orifice located in plane wall perpendicular to the jet axis ($x_0 = 2a_0$).

4. Comparison with controversial data

4.1. *Plane turbulent jets*

As mentioned in §1, there are apparent discrepancies between various experimental data on plane turbulent jets. With the help of the previous analysis it seems possible to unveil the source of the discrepancies.

If $\Theta_w = \pi$, as in Bradbury's (1965) experiments, m vanishes, and (5) provides justification of the classical assumption of constant momentum flux.

For $\Theta_w = \frac{1}{2}\pi$, however, a momentum flux reduction is predicted by (5), which amounts to about 20% at a distance of hundred orifice widths (cf. figure 3). Comparison can be made with experimental data due to Miller & Comings (1957), who defined the jet width such that the data given for jet width and centreplane velocity, respectively, can easily be combined to obtain the momentum flux. The results are shown in figure 3, indicating reasonable agreement between analysis and data. The prediction of the momentum flux reduction is also supported by experimental data obtained by Goldschmidt (1964), Heskestad (1965), Goldschmidt & Eskinazi (1966), and Kotsovinos (1975), as collected by Kotsovinos (1978).

With respect to Hussain & Clark's (1977) investigation of the total momentum flux we observe the following. Let M_t be the total kinematic momentum flux including the pressure integral, i.e.

$$M_t = \int_{-\infty}^{+\infty} (\bar{u}^2 + \overline{u'^2} + \bar{p}/\rho)\, \mathrm{d}Y, \tag{8}$$

where Y is the lateral boundary-layer co-ordinate (with $Y = 0$ being the jet centreplane). In terms of the small parameter ϵ, which characterizes the 'slenderness' of the jet (see (7 *b*)), the contributions from velocity fluctuations and mean pressure, $\overline{u'^2}$ and $\bar{p}$, respectively, are of second order as compared with the first-order term $\bar{u}^2$ (Townsend 1976). Including the mean square of the velocity fluctuations in the momentum balance therefore requires, to be consistent, second-order accuracy in the mean velocity $\bar{u}$. Owing to matching with the induced outer flow (for details see

Appendix A), the second-order mean velocity is negative near the edge of the jet. However, this flow reversal was not taken into account in Hussain & Clark's (1977) total momentum balance. This would explain why a too large total momentum flux was obtained. Further experiments in this direction are desirable.

4.2. *Axisymmetric turbulent jets*

For axisymmetric turbulent jets no data seem to be available which would be suitable for comparisons with the present analysis. Sforza & Mons (1978) found momentum flux variations of less than 9%, but the axial distance was no more than 24 orifice diameters, and wall effects, which are now known to be essential, cannot clearly be identified. According to our table 1, the exponent in (5) is considerably smaller for an axisymmetric jet than for a plane one. Hence it may be difficult to observe the decay effect in axisymmetric turbulent jets.

5. Laminar jet with slowly varying momentum flux

5.1. *Plane laminar jet*

Eliminating $(\mathrm{d}V/\mathrm{d}x)^2$ from (1) and (4), and integrating the differential equation thus obtained, yields

$$\ln\frac{M}{M_\infty} = 6^{\frac{1}{3}} m\, Re_x^{-\frac{2}{3}} \quad (m = \cot\tfrac{1}{3}\Theta_{\mathrm{w}}). \tag{9a}$$

This shows that the kinematic momentum flux M through any cross-section of the plane laminar jet differs but little from its value at infinity M_∞, the relative difference being of the order $O(Re_x^{-\frac{2}{3}})$. Expanding both sides of (9a) for M near M_∞, one finds

$$\frac{M}{M_\infty} = 1 + 6^{\frac{1}{3}} m\, Re_{x,\infty}^{-\frac{2}{3}} + \dots, \tag{9b}$$

where $Re_{x,\infty} = (M_\infty x)^{\frac{1}{2}}\nu^{-1}$. The same result is obtained from a more formal second-order boundary-layer analysis (Mitsotakis *et al.* 1984).

Concerning comparison with experiment, a decrease of momentum flux with increasing distance has already been observed by Andrade (1939) and, more recently, by Sato & Sakao (1964). The latter authors determined M_0/M_∞, i.e. the ratio of the momentum fluxes at the orifice and at infinity, respectively, from data due to Chanaud & Powell (1962). Since the asymptotic analysis ceases to be valid near the orifice, it is insufficient to predict M_0/M_∞ in the required second-order accuracy. Nevertheless, some endorsement can be found (figure 4) by comparing measured values of M_0/M_∞ with predicted values of M/M_∞ at positions where the local Reynolds number, $Re_x = (xM)^{\frac{1}{2}}\nu^{-1}$, is equal to the orifice Reynolds number, $Re_0 = (2a_0 M_0)^{\frac{1}{2}}\nu^{-1}$.

5.2. *Axisymmetric laminar jet*

This case requires a more detailed investigation. In order to adapt the boundary-layer concept for a slow decay of momentum flux, we first define an inner variable ξ_{i} as follows:

$$\xi_{\mathrm{i}} = \tfrac{3}{8} Re_0^2\, \xi, \tag{10}$$

where ξ is defined according to (2), and Re_0 is the Reynolds number of the jet at the orifice, i.e.

$$Re_0^2 = M_0/\nu^2 \gg 1. \tag{11}$$

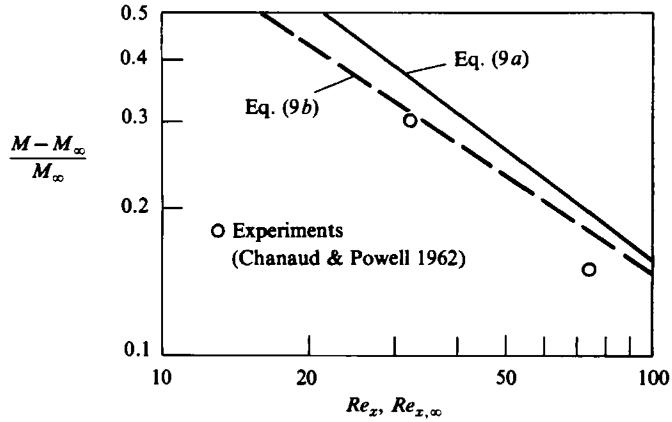

FIGURE 4. Momentum flux versus Reynolds number, for a plane laminar jet emerging from a plane wall perpendicular to the jet axis. (Experimental data for $(M_0 - M_\infty)/M_\infty$ versus Re_0 as given by Sato & Sakao 1964, with an obvious error in Reynolds number corrected according to Zauner 1984.)

Secondly we introduce a 'slowly' varying variable R,

$$R = (r/a_0)^{\epsilon_0}, \tag{12a}$$

$$\epsilon_0 = 16/Re_0^2 \ll 1. \tag{12b}$$

The jet flow is now assumed to depend not only on the original ('fast') radial co-ordinate r but also on the 'slow' variable R (multiple scaling). Separating the variables by writing

$$\psi = 4\nu r f_i(\xi_i, R) \tag{13}$$

for Stokes' stream function ψ, expanding the Navier–Stokes equations for ψ in terms of ϵ_0 with ξ_i fixed, and integrating three times, we obtain

$$\xi_i \frac{\partial f_i}{\partial \xi_i} - f_i + f_i^2 = K_0(R) + K_1(R)\,\xi_i + K_2(R)\,\xi_i^2 \tag{14}$$

with functions of integration $K_0(R)$, $K_1(R)$, and $K_2(R)$. The boundary condition $f_i = 0$ at the axis $\xi_i = 0$ requires $K_0(R) \equiv 0$. Furthermore, matching with the outer expansion, where $f = O(1)$, see (2) and (3), is impossible unless $K_1(R)$ and $K_2(R)$ vanish too. With these simplifications, (14) can be easily integrated. The result is

$$f_i = M(R)\,\xi_i/[M_0 + M(R)\,\xi_i], \tag{15}$$

where $M(R)$ is introduced as a further function of integration. By determining the velocity components from (13) and (15), and integrating with respect to ξ_i, $M(R)$ can be identified as the kinematic momentum flux in the jet. Equation (15) resembles the classical boundary-layer solution (Schlichting 1933, 1979) but with a 'slowly varying' momentum flux $M(R)$ rather than a constant momentum flux $M = M_0$.

The volume flux through a jet cross-section is given by the value of ψ as $\xi_i \to \infty$ and can be found from (13). With $f_i(\infty) = 1$, the entrainment rate $\mathrm{d}V/\mathrm{d}r$ becomes independent of the axial coordinate. This is in agreement with what is also obtained by simply assuming the relation (1), which has been obtained from the classical similarity solution, to be locally valid (with x and r being equivalent). For M cancels in (1) when we substitute for ϵ according to table 1.

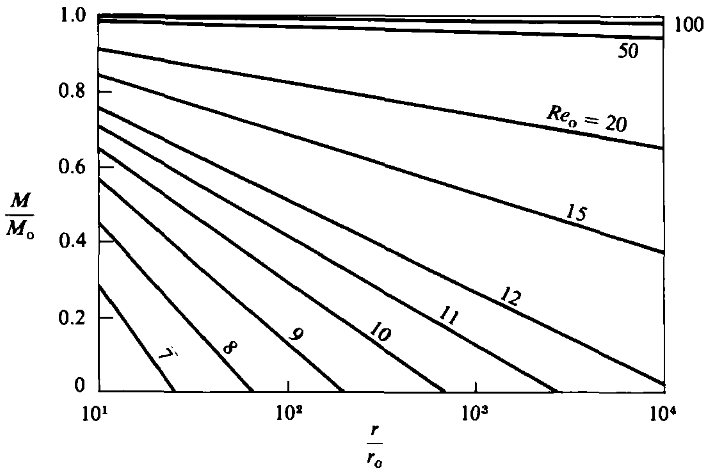

FIGURE 5. Momentum flux in an axisymmetric laminar jet versus distance from the orifice located in an infinite plane wall perpendicular to the jet axis. r_0, constant length of order of orifice radius.

As the entrainment rate is independent of R, the outer solution is the same as it was for constant momentum flux, i.e. it is given by (2) together with the boundary-value problem ($3a$–c) for $f(\xi)$. Furthermore, it follows that (4) can be applied locally to determine the as yet unknown momentum flux $M(R)$. (In a more formal multiple-scaling approach the unknown function $M(R)$ is determined by inspection of the second-order differential equations without recourse to the momentum balance; Appendix B.) Replacing x by r, and introducing R according to ($12a$), one obtains $\mathrm{d}M/\mathrm{d}R = -C^2M_0/2R$. Integration yields

$$\frac{M}{M_0} = 1-\tfrac{1}{2}C^2 \ln\left(\frac{R}{R_0}\right) = 1-\frac{8C^2}{Re_0^2}\ln\frac{r}{r_0}, \tag{16}$$

where the constants of integration R_0 and r_0 have been introduced such that $M = M_0$ at $R = R_0$ and $r = r_0$, respectively. Since M_0 is the momentum flux at the orifice of radius a_0, and M changes but slowly with r, we suppose that $r_0/a_0 = O(1)$, i.e. is independent of Re_0. As with the turbulent case (§3), the value of the constant x_0/a_0 cannot be determined, nor does it matter within the framework of the present asymptotic analysis valid for large distances from the orifice. Simply replacing r_0 by a_0 in (16) causes an error that is of the order of Re_0^{-2}.

Figure 5 displays some results obtained from (16) for a jet emerging from a plane wall perpendicular to the jet axis. Values $r/r_0 < 10$ are not shown, as they are considered too small for the asymptotic analysis to be applicable. It can be seen from figure 5 that the momentum flux is considerably below its value at the orifice for the distances being considered, in particular, if the jet Reynolds number Re_0 is but moderately large. On the other hand, transition to turbulence imposes an upper limit on the Reynolds number. Thus there are severe limitations for applying classical boundary-layer theory, which is based on constant momentum flux, to laminar axisymmetric jets emerging from walls. A critical Reynolds number of about 15 (according to the present definition of Re_0) is predicted by the fully viscous hydrodynamic stability theory (Mollendorf & Gebhart 1973). However, experiments (Reynolds 1962; Hanel & Richter 1979) have shown that axisymmetric jets can remain laminar for fairly long distances (up to a few thousand orifice diameters)

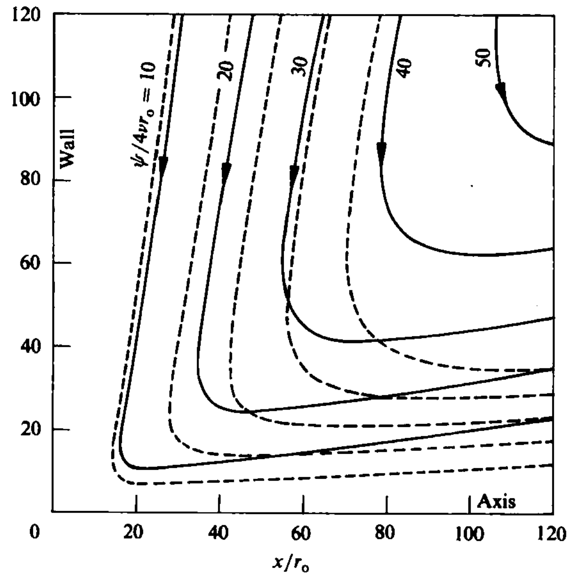

FIGURE 6. Streamlines of an axisymmetric laminar jet flow emerging from an infinite plane wall perpendicular to the jet axis. $Re_0 = 10$. ——, present solution; -----, solution with constant momentum flux (Schneider 1981).

although the Reynolds number is supercritical. This may be explained by the slow growth of weak disturbances in slowly diverging jets (Crighton & Gaster 1976; Plaschko 1979).

Further quantities of interest are the radial velocity component in the jet, $v_r^{(i)}$, and the jet half-angle $\Theta_{\frac{1}{2}}$, i.e. the value of Θ where $v_r^{(i)}$ has dropped to half its value v_m at the axis with r fixed. With $v_r = (1/2r^2)\,\partial\psi/\partial\xi$, one obtains from (13)

$$v_r^{(i)} = \frac{3M}{4\nu r(1+\xi_i)^2}, \tag{17}$$

$$\Theta_{\frac{1}{2}} = 4[2(\sqrt{(2)}-1)/3]^{\frac{1}{2}}\,Re^{-1}, \tag{18}$$

where M and the 'local' jet Reynolds number $Re = M^{\frac{1}{2}}\nu^{-1}$ vary with r according to (16). Hence, in contrast to the classical similarity solution, the velocity at the axis, v_m, decreases more strongly than r^{-1}, and the jet diameter, $2r\Theta_{\frac{1}{2}}$, increases more strongly than r.

Finally, we construct a composite expansion following the multiplicative rule of the method of matched asymptotic expansions. (The additive rule provides a similar result but with the disadvantage of a small error in the non-slip condition at the wall.) Combining $f(\xi)$, which is given by the solution of the boundary-value problem ($3a$–c), and $f_i(\tilde{\xi})$, for which (15) has been obtained, the composite solution becomes:

$$f_\times = f(\xi)\,[1+\tfrac{8}{3}Re_0^{-2}\,\xi^{-1}M_0/M]^{-1}. \tag{19}$$

The r-dependence is contained in M_0/M according to (16).

Streamlines $\psi = 4\nu r f_\times =$ const. are given in figure 6. For the purpose of comparison also shown is the composite solution based on classical boundary-layer theory, i.e. with constant momentum flux in the jet (Schneider 1981). For moderately large distances from the orifice the two solutions differ but little, and in this region the

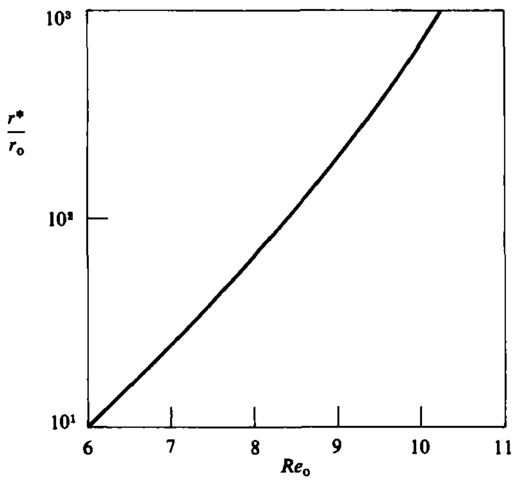

FIGURE 7. Distance of slender-jet termination r^* versus jet Reynolds number Re_0, for an axisymmetric laminar jet ($\Theta_w = 90°$).

theoretical results have already been confirmed by experiments (Zauner 1984, 1985; Schneider 1983). (Note that the streamlines of the outer flow are inclined towards the wall. This is a displacement effect due to the non-slip condition at the infinite wall.) For very large distances, the decay of momentum flux affects the flow field considerably. The distance where the classical solution becomes obsolete depends, of course, on the Reynolds number Re_0, as is quite obvious from (16). For these and other reasons, recent measurements at relatively large Reynolds numbers (Rankin *et al.* 1983) are not conclusive with respect to the present analysis. Experimental results for the proper Reynolds number regime are reported in a companion paper (Zauner 1985), and also the last paragraph of the next section.

6. Termination of the slender jet and the formation of a viscous eddy

While the solution obtained from coupling the jet and the outer flow is well-behaved as $r \to \infty$ if the jet is turbulent, see (5), the corresponding solution for the laminar axisymmetric jet is not. Rather, the momentum flux becomes zero at a certain, finite (if very large) distance r^* from the orifice (figure 5). With $M = 0$, it follows from (17) and (18) that the radial velocity in the jet is zero too, while the lateral extension of the jet becomes infinitely large. Thus, due to the entrainment of momentum from the outer flow, the jet ceases to be slender. For the critical distance r^*, where the termination occurs, one obtains from (16)

$$\frac{r^*}{r_0} = \exp\frac{Re_0^2}{8C^2}. \tag{20}$$

The influence of the half-apex angle of the conical wall, Θ_w, is implicitly contained in the constant C^2, which follows from (3*b*) or table 3. In the limit $\Theta_w \to \pi$ ('vanishing' wall), one obtains $C^2 \to 0$ (Schneider 1981, see also table 3), and $r^*/r_0 \to \infty$. If $\Theta_w < \pi$, however, r^*/r_0 has a finite value that depends enormously on the Reynolds number Re_0 (figure 7).

The *vanishing* momentum flux in the jet is, of course, also in accordance with a global momentum balance for the whole flow field. Since the characteristic length in

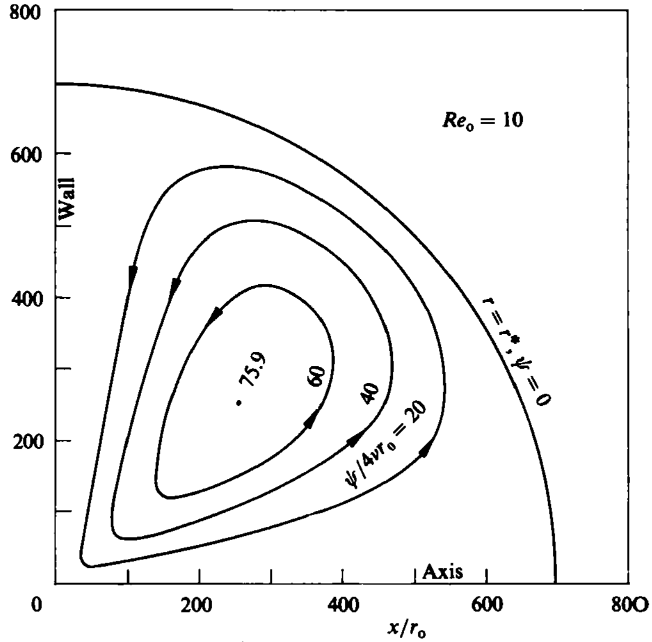

FIGURE 8. Region of closed streamlines of an axisymmetric laminar jet flow emerging from an infinite plane wall perpendicular to the jet axis. (Analysis not strictly valid.)

the outer flow field is of order r and the velocity is of order ν/r, the order of magnitude of both the viscous stresses and the pressure, referred to unit density, is given by $(\nu/r)^2$ (Schneider 1981). Integrating over the wall surface from $r = r_0$ to $r = r^*$ shows that the force exerted on the wall completely balances the momentum source of strength M_0, while the momentum flux through the spherical control surface $r = r^*$ is negligibly small.

Applying the analysis up to the region $r/r^* = O(1)$, where, strictly speaking, it is not valid any more since the slender-jet assumption is violated, we obtain rather peculiar results (figure 8). The composite solution (19) predicts closed streamlines. This recirculatory flow region is completely enclosed by the streamline $\psi = 0$, which is circular in shape and extends from a free stagnation point at the axis ($\xi = 0, r = r^*$) up to the wall. The centre of the toroidal eddy is found by seeking the maximum value of the stream function. With $\psi = 4\nu r f_x$ and (19) one finds the following equations for the co-ordinates ξ_c, r_c of the centre:

$$F^2 + F - 3C^2\xi_c = 0, \tag{21a}$$

$$r_c = r^* \exp\left[(F+1)/3C^2\xi_c\right], \tag{21b}$$

where $F = f(\xi_c)/\xi_c f'(\xi_c)$ is given by the outer-flow solution (Schneider 1981). Note that the angular co-ordinate, ξ_c, as well as the ratio r_c/r^*, are independent of the Reynolds number. For $\Theta_w = 90°$ (plane wall), (21*a*, *b*) yield $\xi_c = 0.146$ ($\Theta_c = 45.0°$) and $r_c/r^* = 0.523$.

Viscous toroidal eddies due to a point source of momentum have already been found by investigating exact solutions of the biharmonic equation (Blake 1971, 1979; Liron & Blake 1981). However, the assumption of creeping flow does not apply to the present problem. Merely for the decay of the viscous eddy at distances much larger than r^* could the biharmonic equation be a valid approximation. This would suggest that the viscous eddy is of infinite extent rather than bounded by a limiting stream surface $\psi = 0$. Details of the decay remain to be studied.

Regarding experiments, it can be seen from (20) and figure 7 that, in a laboratory apparatus of reasonable size, there is only a very limited range of Reynolds numbers which are suitable for observing the termination of the slender jet and the formation of the viscous eddy. This may be the reason why the phenomenon has escaped observation previously. Guided by the present results, however, it is indeed possible to observe the toroidal eddy (Zauner 1985). With respect to the location of the eddy centre, the experiments (Zauner 1985) also confirm the predicted value of Θ_c as well as the variation of r_c according to an exponential function of Re_0^2, yet with a somewhat different value of the constant C^2 in the exponent in (20).

7. Conclusions

Slender jets emerging from orifices in large walls are influenced by entrainment of momentum as follows.

(*a*) *Laminar plane jet.* As the distance from the orifice tends to infinity, the momentum flux approaches a constant, non-zero value that is slightly smaller than the momentum flux at the orifice. The effect on the flow field can be described by second-order boundary-layer theory.

(*b*) *Laminar axisymmetric jet.* The momentum flux changes slowly, yet considerably, when the distance from the orifice increases. There is a strong effect on the flow field, which can be analysed by combining the method of matched asymptotic expansions and the method of multiple scales. When a critical distance is approached, which depends exponentially on the square of the jet Reynolds number, the momentum flux becomes very small, the jet diameter very large, and the analysis breaks down. Together with the termination of the slender jet, the formation of a toriodal viscous eddy is predicted. Experimental confirmation is available (cf. the companion paper by Zauner 1985).

(*c*) *Turbulent (plane or axisymmetric) jets.* The momentum flux vanishes as the distance from the orifice tends to infinity. This is the result of simply coupling jet and induced outer flow via momentum and volume balances, or, more formally, by applying a multiple scaling approach. While jet width or jet diameter are found to grow linearly with axial distance, the axial velocity decreases more rapidly than predicted by classical boundary-layer solutions. This is in agreement with some experimental data, whereas disagreement with other data is believed to be due to the absence of a wall or the failure to take flow reversal into account.

The author is grateful to Mr K. Mitsotakis and Mr E. Zauner for their help in preparing this paper. Mr Mitsotakis also provided some of the numerical values given in tables 2 and 3. The analysis presented in Appendix B was performed by Mr Mörwald as part of his master's thesis.

Appendix A: Asymptotic expansions for plane turbulent jets

Equation (1) defines an entrainment coefficient ϵ. For a self-preserving turbulent jet at large distances from the orifice ($x/a_0 \gg 1$), ϵ is a numerically small constant, (table 1). Thus ϵ may be used as a perturbation parameter, which is equivalent to considering very slender jets, cf. (7*b*).

Dimensionless variables are introduced by referring all lengths (including the Cartesian co-ordinates x, y, figure 1) to the orifice (slit) half-width a_0, the kinematic momentum flux M to its value at the orifice, M_0, the velocity components u and v to $(M_0/a_0)^{\frac{1}{2}}$, and the pressure to $\rho M_0/a_0$, where ρ is the density which is assumed constant. The (dimensionless) pressure difference with respect to the undisturbed pressure at infinity is denoted by p.

It is our aim to find an approximation that is uniformly valid as $x \to \infty$. Classical boundary-layer analysis is unsuitable since it cannot properly take into account the variation of momentum flux (Kraemer 1971; Kotsovinos 1978). A suitable approach is to combine the boundary-layer concept (i.e. co-ordinate stretching with respect to y) and the multiple scaling procedure (with respect to x). Hence we define

$$Y = y/\epsilon, \quad X = x^{\epsilon}, \tag{A 1}$$

and expand the mean variables (denoted by overbar) and the Reynolds stresses ($\overline{u'v'}$, for example) as follows.

Inner expansion:

$$\left.\begin{aligned} \bar{u} &= [M(X)/\epsilon]^{\frac{1}{2}}[\bar{U}_1(x, Y) + \epsilon \bar{U}_2(x, X, Y) + \dots]; \\ \bar{v} &= [\epsilon M(X)]^{\frac{1}{2}}[\bar{V}_1(x, Y) + \epsilon \bar{V}_2(x, X, Y) + \dots]; \\ \bar{p} &= M(X)[\bar{P}_1(x, Y) + \epsilon \bar{P}_2(x, X, Y) + \dots]; \\ \overline{u'v'} &= M(X)[\overline{U'V'}^{(1)}(x, Y) + \epsilon \overline{U'V'}^{(2)}(x, X, Y) + \dots]. \end{aligned}\right\} \tag{A 2}$$

Outer expansion:

$$\left.\begin{aligned} \bar{u} &= [\epsilon M(X)]^{\frac{1}{2}}[u_1(x, y) + \dots]; \\ \bar{v} &= [\epsilon M(X)]^{\frac{1}{2}}[v_1(x, y) + \dots]; \\ \bar{p} &= \epsilon M(X)[p_1(x, y) + \dots]; \\ \overline{u'v'} &= o(\epsilon). \end{aligned}\right\} \tag{A 3}$$

The basic equations are the equation of continuity and the momentum equation, which contains the Reynolds stresses but no molecular stress terms.

Expanding according to (A 3) yields the Euler equations for u_1, v_1 and p_1. This indicates that the outer flow is an inviscid potential flow.

Expanding according to (A 2) yields, in first order,

$$\frac{\partial \bar{U}_1}{\partial x} + \frac{\partial \bar{V}_1}{\partial Y} = 0; \tag{A 4a}$$

$$\frac{\partial \bar{U}_1^2}{\partial x} + \frac{\partial \bar{U}_1 \bar{V}_1}{\partial Y} = -\frac{\partial \overline{U'V'}^{(1)}}{\partial Y}; \tag{A 4b}$$

$$\frac{\partial \bar{P}_1}{\partial Y} = -\frac{\partial \overline{V'^2}^{(1)}}{\partial Y}; \tag{A 4c}$$

i.e. formally the classical boundary-layer equations for free turbulent shear layers (Townsend 1976). For the momentum balance to be considered below, it is important to note that the term $\partial \bar{V}_1^2/\partial Y$ does not appear in the boundary-layer equations as it is small of higher order. This has already been observed by Townsend (1956), but Hussain & Clark (1977) did include the term in their analysis of the total momentum flux.

Boundary conditions are the usual symmetry relations at the centreplane $Y = 0$, and the matching conditions as $Y \to \infty$ and $y \to 0$:

$$\bar{U}_1(x, \infty) = 0, \quad \bar{V}_1(x, \infty) = v_1(x, 0); \tag{A 5a}$$

$$\bar{P}_1(x, \infty) = 0; \tag{A 5b}$$

$$\overline{U'V'}^{(1)} = o(Y^{-1}) \quad \text{as } Y \to \infty. \tag{A 6}$$

(A 6) requires that the velocity fluctuations decay sufficiently fast. This is in agreement with Stewart's (1956) analysis, and is confirmed by measurements (Miller & Comings 1957). (A 5b) may be used to eliminate an unknown function of integration from (A 4c), which yields

$$\bar{P}_1 = -\overline{V'^2}^{(1)}. \tag{A 7}$$

To determine the 'slowly' varying function $M(X)$, the procedure usually applied in a multiple-scaling analysis would be an inspection of the second-order equations to avoid secular terms. This can be done nicely in the laminar flow case, see Appendix B. For the turbulent jet, however, an approach involving the first-order momentum balance seems to be preferable as uncertainties associated with second-order Reynolds stresses are avoided.

Defining the total momentum flux according to (8), introducing dimensionless variables, and expanding according to (A 2) yields

$$M_t = M(X)\left[\int_{-\infty}^{+\infty} \bar{U}_1^2 \, \mathrm{d}Y + \epsilon \int_{-\infty}^{+\infty} (2\bar{U}_1 \bar{U}_2 + \overline{U'^2}^{(1)} + \bar{P}_1) \, \mathrm{d}Y\right]. \tag{A 8}$$

For some applications it could be useful to eliminate the pressure term in (A 8) by means of (A 7), but this is not of interest in the present context.

As the momentum flux changes but slowly with increasing distance from the wall, the first integral on the right-hand side of (A 8) is independent of the 'fast' variable x, i.e. is a constant. Since $M(X)$ is a yet undetermined function, the constant can be given the value 1 without loss of generality, i.e.

$$\int_{-\infty}^{+\infty} \bar{U}_1^2 \, \mathrm{d}Y = 1. \tag{A 9}$$

(A 8) then shows that $M(X)$ is a first approximation of the total kinematic momentum flux. Therefore the momentum balance as outlined in §§2 and 3 can be applied with the following result, which is equivalent to (5):

$$M(X) = (X_0/X)^{\frac{1}{2}m} \quad (X_0 = \text{const}). \tag{A 10}$$

The first-order inner problem as well as the first-order outer problem now resemble their classical counterparts (Tollmien 1926; Taylor 1958) apart from the slowly varying coefficient $M(X)$ in the expansions (A 2) and (A 3). Thus the well-known similarity transformations may be applied, e.g.

$$\bar{U}_1 = x^{-\frac{1}{2}} \frac{\mathrm{d}G}{\mathrm{d}\eta}, \quad \bar{V}_1 = x^{-\frac{1}{2}}\left(\eta \frac{\mathrm{d}G}{\mathrm{d}\eta} - \tfrac{1}{2}G\right), \tag{A 11}$$

where $\eta = Y/x$, and $G(\eta)$ is a reduced stream function which, of course, cannot be determined without modelling – or measuring – the turbulent momentum exchange.

With respect to previous controversies it is important to note that, according to (A 8), both the velocity fluctuation and the mean pressure terms are of second order in the total momentum flux. Matching of the second-order inner expansion and the first-order outer expansion requires

$$\lim_{Y\to\infty} \bar{U}_2 = \lim_{y\to 0} u_1 < 0, \tag{A 12}$$

where the inequality follows from the well-known outer-flow solutions (Taylor 1958). Thus, as already suggested by Kotsovinos (1978), there is a mean flow reversal near the edge of the jet. Recent observations (Goldschmidt, Moallemi & Oler 1983) confirm this prediction. The conclusions regarding momentum flux measurements are given in the main paper (§§4 and 7).

Appendix B: Asymptotic expansions for axisymmetric laminar jets

(*a*) Dimensionless parameters and variables:

$$Re_0 = \frac{M_0^{\frac{1}{2}}}{\nu} \to \infty; \quad \epsilon_0 = \frac{16}{Re_0^2} \to 0; \tag{B 1a}$$

$$\xi = \tfrac{1}{2}(1-\cos\Theta), \quad \tilde{r} = \frac{r}{a_0}, \quad \tilde{\psi} = \frac{\psi}{a_0\,\nu}; \tag{B 1b}$$

$$\tilde{v}_r = \frac{a_0 v_r}{\nu} = \frac{1}{2\tilde{r}^2}\frac{\partial\tilde{\psi}}{\partial\xi}; \tag{B 1c}$$

$$\tilde{v}_\Theta = \frac{a_0 v_\Theta}{\nu} = -\frac{1}{2\tilde{r}}[\xi(1-\xi)]^{-\frac{1}{2}}\frac{\partial\tilde{\psi}}{\partial\tilde{r}}. \tag{B 1d}$$

(*b*) Basic equation: vorticity transport equation:

$$2\xi(1-\xi)\frac{\partial(\tilde{\psi},\tilde{\phi})}{\partial(\tilde{r},\xi)} + \mathrm{E}^4\tilde{\psi} = 0, \tag{B 2a}$$

$$\tilde{\phi} = \frac{\mathrm{E}^2\tilde{\psi}}{4\tilde{r}^2\xi(1-\xi)}, \tag{B 2b}$$

$$\mathrm{E}^2 = \frac{\partial^2}{\partial\tilde{r}^2} + \tilde{r}^{-2}\xi(1-\xi)\frac{\partial^2}{\partial\xi^2} \tag{B 2c}$$

(Milne-Thomson 1968).

(*c*) Boundary conditions:

$$\partial\tilde{v}_r/\partial\xi \text{ bounded}, \quad \tilde{v}_\Theta = 0 \quad \text{on } \xi = 0; \tag{B 3a}$$

$$\tilde{v}_r = 0, \quad \tilde{v}_\Theta = 0 \quad \text{on } \xi = \xi_w; \tag{B 3b}$$

$$\tilde{v}_r = 0, \quad \tilde{v}_\Theta = 0 \quad \text{as } \tilde{r} \to \infty. \tag{B 3c}$$

(*d*) Given momentum flux at $r = r_0 = O(1)$:

$$2\tilde{r}_0^2 \int_0^{\xi_w} \{\tilde{v}_r[(1-2\xi)\,\tilde{v}_r - 2\xi^{\frac{1}{2}}(1-\xi)^{\frac{1}{2}}\,\tilde{v}_\Theta]\}_{\tilde{r}=\tilde{r}_0}\,\mathrm{d}\xi = Re_0^2. \tag{B 4}$$

(*e*) Multiple scaling:

$$\tilde{\psi} = 4\tilde{r}F(\xi, R) \tag{B 5}$$

with

$$R = \tilde{r}^{\epsilon_0}. \tag{B 6}$$

Note that (B 5) assures that there are no secular terms in the velocity components as $\tilde{r} \rightarrow \infty$.

(*f*) First-order inner expansion:

$$F(\xi, R) = F_{io}(\xi_i, R) + O(\epsilon_o) \quad \text{(B 7)}$$

with

$$\xi_i = 6\xi/\epsilon_o \quad \text{(B 8)}$$

yields (cf. §5)

$$F_{io} = \tilde{M}(R)\,\xi_i/[1+\tilde{M}(R)\,\xi_i] \quad \text{(B 9)}$$

with $\tilde{M}(R)$ to be determined from inspection of the second-order equations (see below). With $F_{io} = f_i$, and $\tilde{M}(R) = M(R)/M_o$, (B 9) is equivalent to (15).

(*g*) First-order outer expansion:

$$F(\xi, R) = F_{oo}(\xi, R) + O(\epsilon_o). \quad \text{(B 10)}$$

Matching with first-order inner solutions shows that $F_{oo}(\xi, R) = f(\xi)$ where $f(\xi)$ is to be determined from the boundary-value problem (3*a*–*c*).

(*h*) Second-order inner expansion (Mörwald 1984):

$$F(\xi, R) = F_{io}(\xi_i, R) + \epsilon_o F_{i1}(\xi_i, R) + O(\epsilon_o^2) \quad \text{(B 11)}$$

yields a linear, inhomogeneous differential equation for F_{i1} with general solution

$$\begin{aligned} F_{i1} = -\left[\frac{\xi_i}{6(1+\tilde{M}\xi_i)^2}\right]\Big[& \tilde{M}\xi_i + \tilde{M}^2\xi_i^2 + 2R\frac{\tilde{M}'}{\tilde{M}}\xi_i^{-1} \\ & + C_0(R) + C_1(R)\,(1+\tilde{M}\xi_i)^3\,\tilde{M}^{-1} \\ & + C_2(R)\,(2\ln\xi_i + 4\tilde{M}\xi_i + \tilde{M}^2\xi_i^2) \\ & + C_3(R)\,(-\xi_i^{-1} + 2\tilde{M}\ln\xi_i + \tilde{M}^2\xi_i)\Big]. \end{aligned} \quad \text{(B 12)}$$

With $\tilde{M}$ (from the first order) and $C_0, \ldots C_3$ (from the second order), five functions of the 'slow' variable R are available to satisfy the boundary conditions on the axis and the conditions for matching the second-order inner expansion with the first-order outer expansion. The result is

$$C_1(R) \equiv 0, \quad \text{(B 13}a\text{)}$$

$$C_2(R) = -[1+f'(0)] = C^2 = \text{const}, \quad \text{(B 13}b\text{)}$$

$$C_3(R) = -C^2/\tilde{M}(R), \quad \text{(B 13}c\text{)}$$

and

$$R\tilde{M}'(R) = -\tfrac{1}{2}C^2. \quad \text{(B 13}d\text{)}$$

$C_0(R)$ would have to be determined by an inspection of the third-order equations.

Integration of (B 13*d*) with $\tilde{M} = 1$ at $R = R_0 = \text{const} = O(1)$ yields:

$$\tilde{M}(R) = 1 - \tfrac{1}{2}C^2 \ln(R/R_0). \quad \text{(B 14)}$$

(B 14) is equivalent to (16) which has been obtained in §5 by a different method. Note that $\tilde{M}(R)$, i.e. a slow variation of the momentum flux with $\tilde{r}$, is necessary to eliminate the terms ξ_i^{-1} and $\ln\xi_i$, which produce singularities at the axis $\zeta_i = 0$, in the second-order solution (B 12). This justifies the multiple scaling approach.

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